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Hadron multiplicities in e^+e^- annihilation with heavy primary quarks

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Abstract. The multiple hadron production in the events induced by the heavy primary quarks in e^+e^- annihilation is reconsidered with account of corrected experimental data. A new value for the multiplicity in $b\bar{b}$ events is presented on the basis of pQCD estimates.

1 Introduction

The so-called "naïve model" [1, 2] was the first attempt to give a framework for calculating the multiplicity of hadrons produced in addition to decay products of the heavy quark–antiquark pair in e^+e^- annihilation. Later on, it was argued [3] that the difference between multiplicities in heavy and light quark events ($l = u, d, s$),

$$\delta_{Ql} = N_{Q\bar{Q}}(W) - N_{l\bar{l}}(W), \qquad (1)$$

tends to a constant value at high collision energy:

$$\delta_{Ql} \to \delta_{Ql}^{\mathrm{MLLA}} = 2n_Q - N_{l\bar{l}}(m_Q^2\mathrm{e}). \qquad (2)$$

Here and in what follows, $N_{Q\bar{Q}}$ and $N_{l\bar{l}}$ are mean multiplicities of charged hadrons in heavy and light quark events, respectively.[1]

The comparison with the data has shown that the "naïve model" describes the data on δ_{bl} up to $W = 58\,\mathrm{GeV}$ [1, 4–9], but underestimates the LEP and SLAC data [10–17]. As for the so-called MLLA formula (2), it significantly overestimates both low- and high-energy data on δ_{bl}.

The detailed QCD calculations of the difference between associated multiplicities of charged hadron in e^+e^- annihilation were made in [18, 19]. The QCD expressions for δ_{Ql} from [18, 19] appeared to be in a good agreement with experimental measurements of associated hadron multiplicities in e^+e^- annihilation (see, for instance, [20, 21]). Note that up to now, our formula provided the best description of all the available data on δ_{bl}; see Fig. 1.

Moreover, we made a prediction for δ_{cl} [18, 19]. It is also in very good agreement with all the data on δ_{cl} [1, 5–

7, 12–14]. Let us stress that the very value of δ_{cl} was derived in [18, 19] before the precise measurements of δ_{cl} were made [12–14] that allows to test QCD calculations.[2]

As we will see below, it is the hadron multiplicity in the light quark events that enables one to calculate the multiplicity differences δ_{Ql}. The mean charged multiplicities in $l\bar{l}$ events at different energies corrected for detector effects as well as for initial state radiation were recently cited in [22]. The corrected multiplicity differences averaged over all presently published results have also been presented [22]:

$$\delta_{bl}^{\mathrm{exp}} = 3.12 \pm 0.14, \qquad (3)$$
$$\delta_{cl}^{\mathrm{exp}} = 1.0 \pm 0.4. \qquad (4)$$

The first goal of this paper is to re-estimate our QCD predictions for the quantity δ_{bl}, taking into account the corrected experimental data on $N_{l\bar{l}}(W)$ from [22]. The second goal is to argue that the MLLA formula (2) is nothing but some part of our QCD expression (see Sect. 2), and, as the comparison to the data shows, it should be regarded as a rather rough approximation of the QCD result.[3]

2 QCD formula for multiplicity difference

The hadron multiplicity in a $q\bar{q}$ event, $N_{q\bar{q}}(W)$, looks like [18, 19]

$$N_{q\bar{q}}(Y) = 2n_q + \int\limits_0^Y \mathrm{d}\eta\,\hat{n}_g(Y - \eta)E_q(\eta), \qquad (5)$$

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[1] Everywhere below, it is assumed that we deal with *mean* multiplicities of *charged* hadrons.
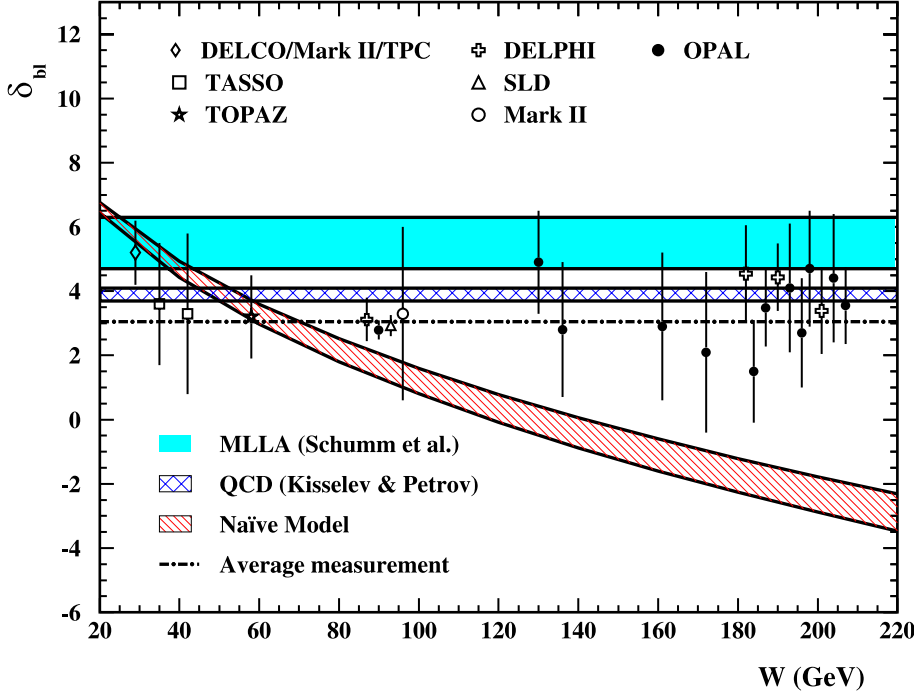
[2] In the low-energy measurements [1, 5–7], the total error of δ_{cl} was about ± 1.5.

[3] The shortcomings of the MLLA formula have already briefly been discussed in [18, 19].

Fig. 1. QCD prediction [18, 19] and MLLA result [3] versus experimental data on the multiplicity difference δ_{bl}. The data are not corrected as in [22]. The prediction of the "naïve model" is also shown

where the variables

$$\eta = \ln \frac{W^2}{k^2} \qquad (6)$$

and

$$Y = \ln \frac{W^2}{Q_0^2} \qquad (7)$$

are introduced. In what follows, the notation $q = Q$ will mean charm or beauty quarks, while the notation $q = l$ will correspond to the massless case (when a pair of u, d- or s-quarks is produced, whose masses are assumed to be zero).

The first term in the r.h.s. of (5), $2n_q$, is the multiplicity of primary hadron decay products. It is extracted from the data ($2n_c = 5.2$, $2n_b = 11.0$ [3], and $2n_l = 2.4$ [20]). The term $E_q(k^2/W^2)$ in (5) is the inclusive spectrum of a gluon jet with a virtuality up to k^2 emitted by primary quarks.[4] It is defined by the discontinuity of the two-gluon irreducible $\gamma^* g^* (Z^* g^*)$ amplitude normalized to the total e^+e^- rate. The quantity $\hat{n}_g(k^2)$ is related to $n_g(k^2)$, the mean multiplicity of hadrons inside this jet:

$$\hat{n}_g(k^2) = \frac{C_F \alpha_s(k^2)}{\pi} n_g(k^2). \qquad (8)$$

Here $\alpha_s(k^2)$ is the strong coupling constant, and $C_F = (N_c^2 - 1)/2N_c$, with N_c being the number of colors.

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[4] It was explained in detail in [18, 19] that one should not consider this mechanism of hadron production via gluon jets as due to "a single cascading gluon", as some authors believe [22]. That E_q is an inclusive spectrum of the gluon jets is seen, e.g., from the fact that $\int (d k^2/k^2) E_q(k^2/W^2) > 1$.

The physical meaning of the function

$$N_q(Y) = \int\limits_0^Y d\eta \, \hat{n}_g(Y - \eta) \, E_q(\eta) \qquad (9)$$

in (5) is the following. It describes the average number of hadrons produced in virtual gluon jets emitted by primary quark (antiquark) of the type q. In other words, it is the multiplicity in a $q\bar{q}$ event except for the multiplicity of the decay products of these quarks at the final stage of hadronization (the first term in (5)).

For the massless case, the function $E \equiv E_l$ was calculated in our paper [18, 19]. In terms of the dimensionless variable
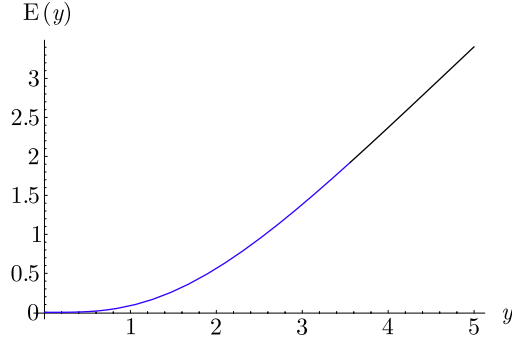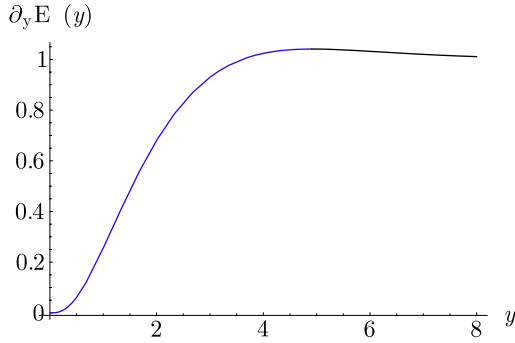
$$\sigma = \exp(-\eta) \qquad (10)$$

it looks like

$$E(\eta(\sigma)) = (1 + 2\sigma + 2\sigma^2) \ln \frac{1}{\sigma} - \frac{3 + 7\sigma}{2}(1 - \sigma)$$
$$- \sigma(1 + \sigma) \left(\ln \frac{1}{\sigma} \right)^2 + 4\sigma(1 + \sigma) I(\sigma),$$
$$\qquad (11)$$

with

$$I(\sigma) = \int\limits_\sigma^1 \frac{d x}{1 + x} \ln \frac{1}{x} \equiv \frac{\pi^2}{4} - \mathrm{Li}_2(1 + \sigma), \qquad (12)$$

where $\mathrm{Li}_2(z)$ is the Euler dilogarithm. The function $E(\eta)$ is presented in Fig. 2. It has the asymptotics $E(\eta)|_{\eta \to \infty} = E^{(\mathrm{asym})}(\eta) = \eta - 1/2$. The derivative of $E(\eta)$ is positive, as one can see in Fig. 3, with $\partial E(\eta)/\partial \eta = 0$ at $\eta = 0$, and

Fig. 2. The function $E(y \equiv \eta)$

Fig. 3. The function $\partial_y E(y \equiv \eta)$

$\partial E(\eta)/\partial \eta = 1$ at $\eta = \infty$. As a result, the associative multiplicity $N_q(W)$ (9) is a monotonically increasing function of the energy W for any positive function $n_g(k^2)$.[5]

Now let us consider the difference between multiplicities in heavy and light quark events, δ_{Ql}, which is defined by (1). The following representation was found in [18, 19]:

$$\delta_{Ql}^{\text{QCD}} = 2(n_Q - n_l) - \Delta N_Q(Y_m). \tag{13}$$

Here, the new notation

$$\Delta N_Q(Y_m) = N_q - N_Q = \int_{-\infty}^{Y_m} dy \, \hat{n}_g(Y_m - y) \Delta E_Q(y), \tag{14}$$
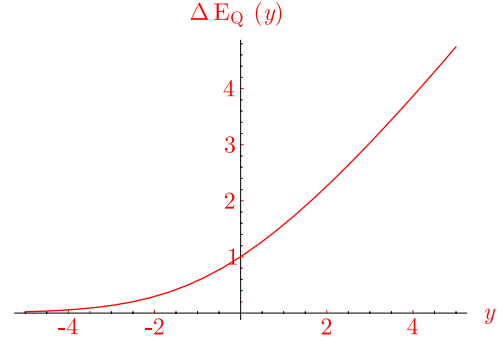
as well as variables

$$y = \ln \frac{m_Q^2}{k^2} \tag{15}$$

and

$$Y_m = \ln \frac{m_Q^2}{Q_0^2} \tag{16}$$

are introduced.

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[5] It results from the relation
$\partial N_q(Y)/\partial Y = \int_0^Y d\eta \, \hat{n}_g(\eta) \, \partial E(Y-\eta)/\partial Y$.

Fig. 4. The function $\Delta E_Q(y)$

The non-trivial result which was obtained in [18, 19] is that the function

$$\Delta E_Q = E_l - E_Q \equiv E - E_Q \tag{17}$$

depends only on the variable

$$\rho = \exp(-y), \tag{18}$$

but not on the energy W. The explicit form of ΔE_Q is known to be

$$\Delta E_Q(y(\rho)) = \left(1 - 3\rho + \frac{7}{2}\rho^2\right)\ln\frac{1}{\rho} + \rho(7\rho - 20)J(\rho)$$
$$+ \frac{20}{\rho - 4}[1 - J(\rho)] + 7\rho + \frac{9}{2}, \tag{19}$$

where

$$J(\rho) = \begin{cases} \sqrt{\frac{\rho}{\rho-4}} \ln\left(\frac{\sqrt{\rho}+\sqrt{\rho-4}}{2}\right), & \rho > 4, \\ 1, & \rho = 4, \\ \sqrt{\frac{\rho}{4-\rho}} \arctan\left(\frac{\sqrt{4-\rho}}{\rho}\right), & \rho < 4. \end{cases} \tag{20}$$

Since $\Delta E_Q(y)$ has the asymptotics

$$\Delta E_Q(y)\Big|_{y\to-\infty} \simeq \frac{11}{3}\exp(-|y|), \tag{21}$$

the integral in (14) converges rapidly at $y \to -\infty$. The function $\Delta E_Q(y)$ is shown in Fig. 4. We find that $\Delta E_Q(y)|_{y\to\infty} = \Delta E_Q^{(\text{asym})}(y) = y - 3/2$.

One should mention the following important relation:[6]

$$\Delta E_Q(y-1) - E(y)\Big|_{y\to\infty} \simeq \frac{5}{2}\sqrt{e}\ln 2 \exp(-y/2). \tag{22}$$

In other words,

$$\Delta E_Q(y) \simeq E(y+1) \tag{23}$$

at large y.

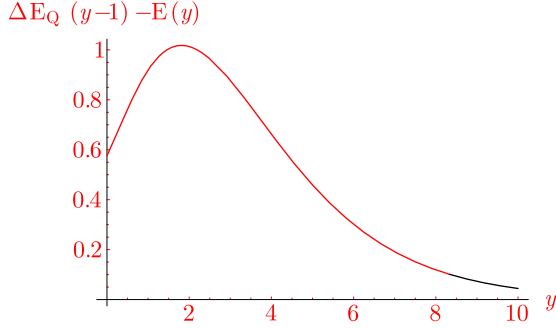[6] Here (and below) e means the base of the natural logarithm.

Fig. 5. The difference $\Delta E_Q(y-1) - E(y)$ as a function of the variable y

If one puts $\Delta E_Q(y) = E(y+1)$, then (neglecting also the contribution from the region $y < -1$)

$$\Delta N_Q = N_{l\bar{l}}(m_Q^2\,\mathrm{e}) - 2n_l\,. \qquad (24)$$

Correspondingly, the approximate expression for δ_{Ql} is then of the form

$$\delta_{Ql}^{(\mathrm{appr})} = 2n_Q - N_{l\bar{l}}(m_Q^2\,\mathrm{e}) = \delta_{Ql}^{\mathrm{MLLA}}\,, \qquad (25)$$

where $\delta_{Ql}^{\mathrm{MLLA}}$ is the MLLA prediction for the multiplicity difference from [3]. Remember that the function $N_{l\bar{l}}(W)$ describes the hadron multiplicity in a light quark event at colliding energy W.

However, (23) is very far from being satisfied at relevant $y < Y_m$,[7] as it is clearly seen in Fig. 5. As a result, there could be a large difference between $\delta_{Ql}^{(\mathrm{appr})}$, see (25), and the QCD expression $\delta_{Ql}^{\mathrm{QCD}}$, see (13).

To demonstrate this, it is convenient to represent the expression for ΔN_Q (14) in the form

$$\Delta N_Q(Y_m) = \int_0^{Y_m+1} \mathrm{d}y\,\hat{n}_g(Y_m+1-y)E(y)$$

$$+ \int_{-\infty}^{-1} \mathrm{d}y\,\hat{n}_g(Y_m-y)\Delta E_Q(y)$$

$$+ \int_0^{Y_m+1} \mathrm{d}y\,\hat{n}_g(Y_m+1-y)[\Delta E_Q(y-1) - E(y)]$$

$$\equiv \left[N_{l\bar{l}}\left(m_Q^2\,\mathrm{e}\right) - 2n_l\right] + \delta N_Q^{(1)}(Y_m) + \delta N_Q^{(2)}(Y_m)\,, \qquad (26)$$

that results in the formula (see (13))

$$\delta_{Ql}^{\mathrm{QCD}} = 2n_Q - N_{l\bar{l}}(m_Q^2\,\mathrm{e}) - \delta N_Q^{(1)}(Y_m) - \delta N_Q^{(2)}(Y_m)$$

$$= \delta_{Ql}^{(\mathrm{appr})} - \delta N_Q^{(1)}(Y_m) - \delta N_Q^{(2)}(Y_m)\,. \qquad (27)$$

Here we have introduced the notation

$$N_Q^{(1)}(Y_m) = \int_{-\infty}^{-1} \mathrm{d}y\,\hat{n}_g(Y_m-y)\Delta E_Q(y) \qquad (28)$$

and

$$N_Q^{(2)}(Y_m) = \int_0^{Y_m+1} \mathrm{d}y\,\hat{n}_g(Y_m+1-y)[\Delta E_Q(y-1) - E(y)]\,. \qquad (29)$$

Note that both $N_Q^{(2)}(Y_m)$ and $N_Q^{(2)}(Y_m)$ are positive functions, since $\Delta E_Q(y) > 0$ at all y and $\Delta E_Q(y-1) - E(y) > 0$ at $y \geqslant 0$ (see Figs. 3 and 4).

In order to exploit the corrected data on $N_{l\bar{l}}(W)$ at $W = 8$ GeV,

$$N_{l\bar{l}}(8.0\,\mathrm{GeV}) = 6.70 \pm 0.34\,, \qquad (30)$$

we have chosen the mass of the b-quark to be $m_b = 4.85$ GeV, which corresponds to $m_b\sqrt{\mathrm{e}} = 8$ GeV.

The estimates show that the dominant correction to $\delta_{Ql}^{\mathrm{QCD}}$ is $\delta N_Q^{(2)}$, not $\delta N_Q^{(1)}$. To calculate a lower bound of $\delta N_b^{(2)}$, let us use the following inequality:

$$\Delta E_Q(y) \geq E(y + \Delta y_Q)\,. \qquad (31)$$

Note that Δy_Q is a monotonically non-increasing function of $y \geqslant 0$ and that it tends to 1 at large y. It solves the equation

$$\Delta E_Q(Y_m) = E(Y_m + \Delta y_Q)\,, \qquad (32)$$

where Y_m is defined above (16). Then we get from (29) and (31)

$$\delta N_Q^{(2)} \geqslant N_{l\bar{l}}(Y_m + \Delta y_Q) - N_{l\bar{l}}(Y_m + 1)$$

$$- \int_0^{\Delta y_Q - 1} \mathrm{d}y\,\hat{n}_g(Y_m + \Delta y_Q - y)E(y)\,. \qquad (33)$$

For our further estimates, we need to know the hadron multiplicity in light quark events in the energy interval $2.5\,\mathrm{GeV} \leqslant W \leqslant 29\,\mathrm{GeV}$. By fitting the data on the hadron multiplicity in the light quark events at low W, we get the expression:

$$N_{l\bar{l}}(W) = 2.07 + 1.11\ln W + 0.54\ln^2 W\,. \qquad (34)$$

Putting $Q_0 = 1$ GeV, we find $\Delta y_b = 1.61$. Taking into account that the last term in (33) is negligible,[8] we get from (33) and (34)

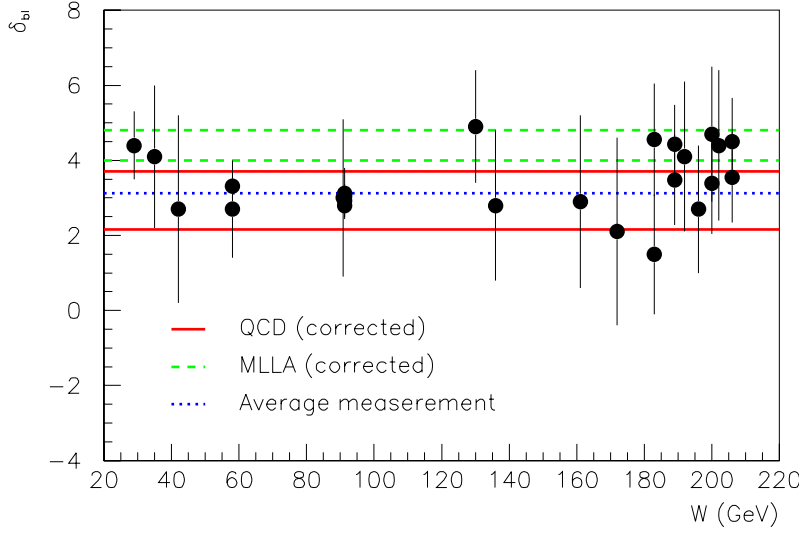$$\delta N_b^{(2)} \geqslant 1.07\,. \qquad (35)$$

Fig. 6. Our new QCD result for δ_{bl} (the corridor between the two solid lines) as well as MLLA prediction from [22] (the corridor between the two dashed lines) versus *corrected* experimental data

Correspondingly, our prediction accounting for the revision of the data on the multiplicity in the light quark events,

$$\delta_{bl}^{\mathrm{QCD}} \leqslant 2n_b - N_{l\bar{l}}(Y_m + \Delta y_b) = 3.33 \pm 0.38 \,, \quad (36)$$

appears to be lower than our previous result $\delta_{bl} = 3.68$ [18, 19]. We have used the value

$$2n_b = 11.10 \pm 0.18 \,. \quad (37)$$

The error of $N_{l\bar{l}}$ was taken to be ± 0.34. Let us stress that our upper bound (36) is very close to the present experimental value of $\delta_{bl}^{\mathrm{exp}}$ in (3).

Now let us derive a lower bound on $\delta_{bl}^{\mathrm{QCD}}$. To do this, let us start from (14). It is convenient to represent the integral in (14) as a sum of two terms:[9]

$$\Delta N_b = \int\limits_{-4}^{-1} \mathrm{d}y\, \hat{n}_g(Y_b - y) \Delta E_b(y) + \int\limits_{-1}^{Y_b} \mathrm{d}y\, \hat{n}_g(Y_m - y) \Delta E_b(y)$$

$$= \Delta N_b^{(1)} + \Delta N_b^{(2)} \,, \quad (38)$$

with $Y_b = \ln(m_b^2/Q_0^2) \simeq 3.16$. Consider the first term in (38). One can check that

$$\Delta E(y) < 0.18 E(y + 5.8) \quad (39)$$

in the region $-4 < y < -1$, which leads to the inequality

$$\Delta N_b^{(1)} < 0.18 \int\limits_{1.8}^{4.8} \mathrm{d}y\, \hat{n}_g(Y_b + 5.8 - y) \Delta E_b(y) \,. \quad (40)$$

The estimations show that $\hat{n}_g(Y_b + 5.8 - y) < 2\hat{n}_g(4.8 - y)$ when y varies from 1.8 to 4.8. Thus, we get

$$\Delta N_b^{(1)} < 0.36 \left[N_{l\bar{l}}(W = 11 \text{ GeV}) - N_{l\bar{l}}(W = 2.5 \text{ GeV})\right]$$

$$= 1.54 \pm 0.17 \,. \quad (41)$$

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[9] We took into account that the region $-\infty < y < -4$ gives a negligible contribution to ΔN_b.

The second term in (38) can be estimated by using the inequality

$$\Delta E(y) < 0.62 E(y + 3.5) \,, \quad (42)$$

which is valid in the region $-1 < y < Y_b$. Then

$$\Delta N_b^{(2)} < 0.62 \left[N_{l\bar{l}}(W = 28 \text{ GeV}) - N_{l\bar{l}}(W = 3.5 \text{ GeV})\right]$$

$$= 4.61 \pm 0.30 \,. \quad (43)$$

As a result, we obtain from (12), (14) and (41), (43) the following lower bound on $\delta_{bl}^{\mathrm{QCD}}$:

$$\delta_{bl}^{\mathrm{QCD}} > 2.55 \pm 0.39 \,. \quad (44)$$

Figure 6 demonstrates that our QCD predictions for δ_{bl} are very close to the corrected experimental data.

Our results can be compared with the MLLA expectation reported recently in [22]:

$$\delta_{bl}^{\mathrm{MLLA}} = 4.4 \pm 0.4 \,. \quad (45)$$

Note that the scheme of [22] is not stable against next-to-MLLA corrections. According to A(30) from [22], the MLLA prediction (25) is modified as follows:

$$\delta_{Ql}^{\mathrm{NMLLA}} = 2n_Q - N_{l\bar{l}}(m_Q^2\, \mathrm{e})$$

$$\times \left\{ 1 + \frac{3\alpha_{\mathrm{s}}(m_Q)}{2\pi} \left[\frac{\pi^2}{24} + \left(\frac{\pi^2}{3} - \frac{5}{4} \right) \right] \right\} \,. \quad (46)$$

The next-to-MLLA corrections in (46) change the result (45) to

$$\delta_{bl}^{\text{NMLLA}} = 2.6 \pm 0.4 \,. \tag{47}$$

The situation is worse in the case of a c-quark. The formula (46) results in a unsatisfactory low value:

$$\delta_{cl}^{\text{NMLLA}} = -0.1 \pm 0.4 \,. \tag{48}$$

This demonstrates us once more that the lowest-order MLLA expression (2) is not correct.

Moreover, as we have shown above, the deviation of the function $\Delta E_Q(y)$ from $\Delta E_Q^{(\text{asym})}(y) = y - 3/2$, as well as the deviation of $E(y)$ from $E^{(\text{asym})}(y) = y - 1/2$, cannot be neglected. In other words, the MLLA formula (25) is, in fact, not a full QCD result.[10] It is nothing but *a part* of the correct QCD formulae (1) and (14) in a very rough approximation, $E(y) = \Delta E_Q(y-1)$. So it is senseless to try to "improve" it with next-to-MLLA calculations.

3 Conclusions

We have derived the QCD formula for the difference between hadron multiplicities in heavy and light quark events in e^+e^- annihilation (with Q being a type of a heavy quark):

$$
\begin{aligned}
\delta_{Ql}^{\text{QCD}} = {} & 2n_Q - N_{l\bar{l}}(m_Q^2 \mathrm{e}) \\
& - \int_{Q_0^2}^{m_Q^2 \mathrm{e}} \frac{\mathrm{d}k^2}{k^2} \hat{n}_g(k^2) \left[\Delta E_Q\left(\frac{m_Q^2}{k^2}\right) - E\left(\frac{m_Q^2 \mathrm{e}}{k^2}\right) \right] \\
& - \int_{m_Q^2 \mathrm{e}}^{\infty} \frac{\mathrm{d}k^2}{k^2} \hat{n}_g(k^2) \Delta E_Q\left(\frac{m_Q^2}{k^2}\right) \,.
\end{aligned}
\tag{49}
$$

Here $n_g(k^2)$ describes the mean number of charged hadrons in the gluon jet with the virtuality up to k^2, and E, ΔE_Q are known functions.

By using the data on the hadron multiplicity in light quark events $N_{l\bar{l}}$, corrected for the detector effects and initial state radiation effects [22], we have obtained from (49) the bounds

$$2.2 < \delta_{bl}^{\text{QCD}} < 3.7 \,. \tag{50}$$

Let us note that this estimate does not depend on a specific choice of the function $n_g(k^2)$, and it is in a good agreement with the average experimental value $\delta_{bl}^{\text{exp}} = 3.12 \pm 0.14$.

The two last terms in (49) are positive and numerically large.[11] As a result, a deviation of the MLLA prediction,

$$\delta_{bl}^{\text{MLLA}} = 2n_b - N_{l\bar{l}}(m_b^2 \exp) \,, \tag{51}$$

from the QCD expression,[12]

$$\delta_{bl}^{\text{QCD}} = 2(n_b - n_l) - \int_{Q_0^2}^{m_b^2} \frac{\mathrm{d}k^2}{k^2} \hat{n}_g(k^2) \Delta E_Q\left(\frac{m_b^2}{k^2}\right) \,, \tag{52}$$

appears to be significant.

As one can see, the MLLA formula is too rough an approximation of the QCD formula. The former results from the latter on the assumption that the quantities $E(y)$ and $\Delta E_Q(y)$ can be replaced by their asymptotics at $y \to \infty$. Since the relevant values of y are far from being very large, this assumption is not correct, and it leads to a significant overestimation of δ_{bl}. Thus, any attempt to use the MLLA expression (2) as a first-order approximation for higher-order calculations (as it is done in [22]) is poorly justified.

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[10] It explains why formula (45) overestimates the data by more than one unit.

[11] In particular, the second term in (49) (dominating the third one) is equal to 1.1 for the case of the beauty pair production ($m_Q = m_b$, $n_Q = n_b$).

[12] This formula is an equivalent compact form of (49) for $Q = b$.

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